LA-UR-80-591

TITLE: FUN WITH E_6

MASTER

AUTHOR(S): Richard Slansky

SUBMITTED TO: Proceedings of the Coral Gables Orbis Scientiae, 1980
CORAL GABLES, FLA., JANUARY 14-17, 1980

DISCLAIMER

University of California

LOS ALAMOS SCIENTIFIC LABORATORY
Post Office Box 1663 Los Alamos, New Mexico 87545
An Affirmative Action/Equal Opportunity Employer

Form No. 836 R3
St. No. 2629
12/78

UNITED STATES
DEPARTMENT OF ENERGY

FUN WITH E_6

R. Slansky
Theoretical Division, Los Alamos Scientific Laboratory*
University of California, Los Alamos, New Mexico 87545

ABSTRACT

The exceptional Lie group E_6 is a candidate local symmetry for a Yang-Mills theory that unifies electromagnetic, weak, and strong interactions. Several ways of incorporating the fermion spectrum are discussed, including an amusing example where some of the known spin 1/2 fermions are composite states of elementary fermions and some scalar particles in the theory. The symmetry properties and the representations of E_6 are reviewed, the symmetry breaking classified, and the dynamical breaking of the weak interaction gauge group is discussed, all in some detail using Dynkin's representation theory.

---

*Work supported by the U. S. Department of Energy.

To be published in the Proceedings of the Coral Gables Orbis Scientiae, 1980.

1. INTRODUCTION

It does not look likely that it should be much fun to explore the features of a Yang-Mills theory based on a group of rank 6 with 78 generators. Just writing down the commutation relations of the exceptional Lie algebra E_6 in a useful basis looks like a mess, and working with representations of dimension 27, 78, 351, or higher would seem to be, at best, tedious. The only way to have any fun at all in exploring the consequences of such a big group is to have an easily applied technology. In fact, Dynkin's representation theory[1] is quite adequate for obtaining many of the symmetry results of the theory. The purpose of this talk is twofold: (1) to "review" how these techniques can be used to explore certain consequences of Yang-Mills theories: and (2) to apply these techniques to an E_6 model in which the scalars in an adjoint $\underline{78}$ form composite bound states with the elementary fermions in a $\underline{27}$ to give a total of three quark-lepton families. Much of the research reported here was done in collaboration with Gordon Shaw.[2] I have written up these notes in an order reversed from the talk at Orbis Scientiae and the presentation of Ref. 2, because I would like to use this opportunity to emphasize the usefulness of Dynkin's representation theory for exploring unified models. Instead of starting with the physics of the composite fermion model, which the reader will find in Sec. 3, I will review some of the mathematics of E_6 first, beginning with a rapid review of Dynkin's representation theory. We can then use the language developed in Sec. 2 with impunity for describing the composite fermion model.

2. SYMMETRY PROPERTIES OF E_6

Our object in this Sec. is to review the quantum number structure of the representations of E_6 in a fashion that is very convenient for many other applications. This means that we must be able to identify the physical significance and compute the eigenvalues of the 6 diagonalizable generators of E_6, when acting on vectors in the representation, and then learn what the 72 ladder operators do to these quantum numbers. The Dynkin formalism works for any simple algebra. It can be viewed as a fancy way for keeping track of the quantum numbers, and as such, it is merely a mathematical bookkeeping trick. However, it does greatly simplify doing the physics too, and I hope the reader will find it to be fun.

The rank of an algebra G is the number of independent diagonalizable generators in G; these ℓ generators form the Cartan subalgebra of G. The remaining generators can be written as ladder operators, which, when acting on a Hilbert space vector in a representation, change the set of ℓ eigenvalues by amount α. The ℓ eigenvalues correspond to a point in an ℓ dimensional Euclidean space, which is called the weight of the representation vector Λ. The root α is also a vector with ℓ components in weight space, and it represents a permissible shift from one weight to another. If Λ and $\Lambda+\alpha$ are both weights of a unitary irreducible representation (irrep) of G, then the ladder operator E_α acting on $|\Lambda\rangle$ is proportional to $|\Lambda+\alpha\rangle$. The correspondance between weight space and representation space is a key point in representation theory.[1,3]

A convenient basis for the weight space is formed by the ℓ simple roots of the algebra; these specially chosen roots have the relative lengths and angles indicated by the Dynkin diagram.[1] The simple root basis is not an orthonormal basis for weight space, but otherwise, it is extremely useful. Each representation vector is labeled by a weight. In the "Dynkin basis" the components of a weight Λ are ℓ integers a_i, defined by the scalar products,

$$a_i = \frac{2(\Lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \quad , \; i = 1,2,\ldots,\ell, \qquad (1)$$

where α_i is the i-th simple root. The p. f that a_i must be an integer is a generalization of the proof that the magnetic quantum number of an SU_2 vector must be integer or half integer. (For SU_2, $a_1 = 2m$.) The computation of the set of weights of an irrep written in the Dynkin basis $(a_1 a_2 \ldots a_\ell)$ is straightforward. There exists one weight, the highest weight, that always labels a unique Hilbert space vector in the irrep. Each set of ℓ nonnegative integers uniquely gives the highest weight of an irrep of G, and these sets exhaust the entire set of finite dimensional irreps. Starting from the highest weight, the remaining weights are computed by subtracting off simple roots: if at any level (the level is the number of simple roots that have been subtracted from the highest weight) the a_i coefficient is positive, then the i-th simple root can be subtracted off a_i more times, and the resulting weight is a weight in the irrep. The maximum number of simple roots that can be subtracted off the highest weight is the scalar product of the highest weight and the level vector; the level vector for E is $\tilde{R}$ = [16,30,42,30,16,22]. (The computation of scalar products will be considered shortly.) Moreover, the weight system of the representation must be spindle-shaped; the number of weights at level k is equal to the number of weights at level $\tilde{R}\cdot\Lambda - k$ (Λ the highest weight), and the number of weights at level k+1 is $\geq$ to that at level k, for k less than $\tilde{R}\cdot\Lambda/2$. A given weight may be obtained by several routes. The degeneracy of a weight can be computed (with a bit more difficulty) from the Freudenthal recursion relation, but for the low lying irreps studied here, the degeneracy is easily guessed. (The degeneracy is the number of Hilbert space vectors in an irrep with the same weight; additional labels are needed to distinguish degenerate weights.)

We will need to compute many scalar products in weight space. Because the simple root basis is not orthonormal, the scalar product of weights with components a_i and a'_i involves a metric tensor, which is closely related to the inverse of the Cartan matrix A^{-1}, and is A^{-1} for the algebras where the simple roots all have the same lengths. Thus, the scalar product of Λ and Λ' is

$$(\Lambda, \Lambda') = \sum_{i,j=1}^{\ell} a_i (A^{-1})_{ij}\, a'_j \equiv \tilde{\Lambda}\cdot\Lambda = \sum_{i=1}^{\ell} \tilde{a}_i\, a'_i \,, \qquad \tilde{a}_i = a_j (A^{-1})_{ji} \,. \tag{2}$$

We will often give Λ in the "Dynkin basis" $(a_1 \ldots a_\ell)$, and we will call it $\tilde{\Lambda}$ when multiplied by the metric, $\tilde{\Lambda} = [\tilde{a}_1 \ldots \tilde{a}_\ell]$, putting the components in square brackets. The inverse of the Cartan matrix (A itself can be read off the Dynkin diagram) for E_6 is

$$A^{-1} = \frac{1}{3}\begin{pmatrix} 4 & 5 & 6 & 4 & 2 & 3 \\ 5 & 10 & 12 & 8 & 4 & 6 \\ 6 & 12 & 18 & 12 & 6 & 9 \\ 4 & 8 & 12 & 10 & 5 & 6 \\ 2 & 4 & 6 & 5 & 4 & 3 \\ 3 & 6 & 9 & 6 & 3 & 6 \end{pmatrix} \tag{3}$$

Let us apply these results to some E_6 irreps. The representation with highest weight (1 0 0 0 0 0) is complex and $\underset{\sim}{27}$ dimensional. With Dynkin's conventions, the first simple root is (2-1 0 0 0 0), so the first level weight is then (-1 1 0 0 0 0); the second level is (0-1 1 0 0 0); the third level is (0 0-1 1 0 1); the fourth level has two weights, (0 0 0-1 1 1) and (0 0 0 1 0-1); and so on to the 16-th level, (0 0 0 0-1 0). The $\overline{\underset{\sim}{27}}$ has highest weight (0 0 0 0 1 0), and the adjoint $\underset{\sim}{78}$ has highest weight (0 0 0 0 0 1).

We shall show shortly that in the usual embedding of color and flavor in E_6 , the electric charge operator, which is in the Cartan subalgebra, is measured along the axis,

$$\tilde{Q}^{em} = \frac{1}{3}[\ 2\ 1\ 2\ 0\ 1\ 0], \tag{4}$$

where the normalization of $\tilde{Q}^{em}$ is chosen so that the electric charge of any weight (or state) is given by the scalar product,

$$Q^{em}_{\Lambda} = \tilde{Q}^{em} \cdot \Lambda \ . \tag{5}$$

Thus, the (1 0 0 0 0 0) has electric charge 2/3, (-1 1 0 0 0 0) has charge -1/3, (0 0-1 1 0 1) has charge -2/3, and so on. We will need to discuss the conventions concerning the embedding of electromagnetism in E_6.

Before we can identify the physical relevance of the roots and axes in weight space, we must find out how color and flavor are embedded in E_6. This embedding can be done in a coordinate independent fashion.[4,5] There is only one embedding of QCD and QED that seems to have a chance of being relevant. This embedding is identified by the requirement that the 27 has 9 color singlets, 3 quarks and 3 antiquarks, where two of the quarks have electric charge -1/3 and the other quark has charge 2/3. (The embeddings with one singlet, one octet, 3 quarks, and 3 antiquarks appear irrelevant, as do more exotic charge assignments.) For many purposes this coordinate independent statement of the embedding is sufficient. However, for practical calculations of symmetry breaking and mass matrices, it is often helpful to have a coordinatization of the weight space. We follow a certain set of conventions here; other conventions are related by a Weyl reflection.

The possible subgroup chains that lead to these color and charge assignments have been classified by Dynkin.[1] The most useful for physics are

$$E_6 \supset SO_{10} \supset SU_5 \supset SU_2^{\,W} \times SU_3^{\,c} \quad , \tag{6}$$

$$E_6 \supset SO_{10} \supset SU_2^{\,W} \times SU_2 \times SU_4 \quad (SU_4 \supset SU_3^{\,c}) \ , \tag{7}$$

$$E_6 \supset SU_3 \times SU_3 \times SU_3^{\,c}, \tag{8}$$

where the first SU_3 in (8) contains SU_2^w, the weak isospin. (We have ignored the U_1's, but this omission will be filled in later.) Our embedding conventions are to follow Ref. 6, where the highest weights of an irrep are projected onto highest weights of the irreps to which it branches, for the chain in (6). Then for (7) and (8), we require that the same physical directions in E_6 weight space as derived from (6) are maintained for the other embeddings. (For example, the same roots correspond to SU_3^c, etc.) The projection matrices for the subgroup chain in (6) are[6]

$$P(E_6 \supset SO_{10}) = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (9)$$

$$P(SO_{10} \supset SU_5) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \quad (10)$$

$$P(SU_5 \supset SU_2 \times SU_3) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad (11)$$

The projection matrix is not square if there is a loss of rank or a U_1 factor in going to the subgroup. The matrix elements are nonnegative integers because of the conventions followed.

We identify the SU_2 and SU_3 of (11) as the weak and color groups, and use this subgroup chain to identify the physical significance of the roots of E_6. This identification is worked out in Table 1. Consider the example of the E_6 root (1-1 1-1 1 0), which is projected onto (-1 0 1 0 0) by (9). That is a root in the $\underline{45}$ of SO_{10}. This SO_{10} weight is then projected by (10) to the SU_5 weight (-1 1 0 1), which is a root in the adjoint $\underline{24}$. Finally (11) projects (-1 1 0 1) to (1)(0 1) of $SU_2 \times SU_3$, which identifies the (1-1 1-1 1 0) root of E_6 as a color triplet with $I_3^W = 1/2$; it is the charge 4/3, SU_5 antilepto-diquark that mediates proton decay. It is a simple computation to construct the rest of Table 1 for the $\underline{78}$, and also to work out Table 2 for the $\underline{27}$ of E_6. The columns labeled $SU_5(SO_{10})$ gives the SO_{10} irrep into which the E_6 weight branches, and then tells which SU_5 irrep that the SO_{10} weight branches into.[7] The simple roots are recovered at level 10 or minus level 12.

In computing the projection matrices for the subgroup chains (7) and (8), we require that the E_6 roots have the same interpretation as the chain given by (9), (10), and (11). That is our convention. Thus we must require that the projection matrix for (8) carries the (1-1 1-1 1 0) root to (1 0)(1 0)(0 1) of $SU_3 \times SU_3 \times SU_3^c$, since this root has $I_3^W = 1/2$, $Q^{em} = 4/3$, and is an antiquark. Since the $\underline{78}$ branches to $(\underline{8},\underline{1},\underline{1}) + (\underline{1},\underline{8},\underline{1}) + (\underline{1},\underline{1},\underline{8}) + (\underline{3},\underline{3},\underline{\bar{3}}) + (\underline{\bar{3}},\underline{\bar{3}},\underline{3})$, we see that the identification is unique. For the subgroup chain in (7), we use (9) and find the projection matrices,

$$P(SO_{10} \supset SU_2 \times SU_2 \times SU_4) = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 0 \end{pmatrix} \tag{12}$$

$$P(SU_4 \supset SU_3^c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{13}$$

Of course, (12) and (13) are useful for studying SO_{10} theories. Finally, we find for the projection matrix of (8),

$$P(E_6 \supset SU_3 \times SU_3 \times SU_3{}^c) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}. \qquad (14)$$

For SU_5 theories, this machinery is stronger than is usually needed, although it is very easy to use and actually simplifies many computations. (As an exercise, consider the SU_5 fermion mass matrix including the $\underline{45}$ and $\underline{50}$ terms: remember to use the Kronecker products in doing this.)

The eigenvalues of the generators in the Cartan subalgebra can be const: .ed and computed in a straightforward fashion by computing the axes and normalization as done for the electric charge in (4). Since the values of these quantum numbers are already known from coordinate independent methods,[5] the computation is well defined. In Table 3 we have listed a complete set of generators for the flavor interactions. Two of the Cartan subalgebra members are in $SU_3{}^c$, and the other 4 include: the U_1 in $E_6 \supset SO_{10} \times U_1$, which we call $U_1{}^t$; the U_1 in $SO_{10} \supset SU_5 \times U_1$, which we call $U_1{}^r$; and the weak isospin and hypercharge axes. These latter axes are normalized in the usual way, so $Q^{em} = I_3{}^w + Y^w/2$.

The quantum number structure of the E_6 generators and irreps is summarized in Tables 1, 2, and 3. In Table 1, half of the nonzero roots of E_6 are listed; the negatives of those listed and the zero roots are not listed. The $\underline{27}$ is complex, so the negative of a weight in the $\underline{27}$ is not in the $\underline{27}$; it is in the $\overline{\underline{27}}$. Table 3 contains projection axes and the weights of several important roots. I hope that the reader will enjoy checking these tables and finding just how simple it is to identify the physical content of the E_6 generators and states. These techniques can be used for even bigger groups, such as SO_{18} or SO_{22}, and for even larger irreps, such as the <u>351</u> or <u>1728</u> of E_6 [8] without using a computer, although a program does exist.[6]

We now turn to an important application of the above formalism, that of investigating the symmetry breaking of E_6. Instead of trying to set up a Higgs potential and doing a messy minimization problem, we apply several physical constraints that any symmetry breaking mechanism must satisfy if the standard model is to be recovered in its usual form.

If a field or "effective field" has a nonzero vacuum expectation value, then the weight of that field determines much about the symmetry breaking. E_6 models have a set of lepto-diquark bosons that mediate proton decay in second order, so these bosons must have superlarge ($\gtrsim 10^{14}$ GeV) masses from a big symmetry breaking, just as in the SU_5 model.[9] The direction of the big breaking in the 6-dimensional weight space, written in the Dynkin basis (1), is called B. Similarly, the little breaking, which gives masses to the weak bosons and may also contribute to other bosons in the theory, is called L. Of course there may be some intermediate mass scales, but generalizations to that case will be obvious.

The weight-space direction of B must be perpendicular to the SU_3^c roots, or else the color-changing gluons will acquire masses, and it must be perpendicular to I_+^W, the root of the weak isospin raising operator, so that the charged, weak boson does not get a superlarge mass. In Table 1 we find that the SU_3^c roots lie in the plane formed by (0 0 0 0 0 1) and (0 1 0 0-1 0), which implies that all symmetry breaking directions must have the parameterization, $\tilde{B} = [-c\ d\ a\ b\ d\ 0]$, where $\tilde{B}$ is defined in ([illegible]). The I_+^W weight is (1 0 0 0 1-1), so the big breaking has the form,

$$\tilde{B} = [-c\ c\ a\ b\ c\ 0] \quad . \tag{15}$$

The column labeled $\tilde{B}\cdot\alpha$ in Table 1 gives a parameterization of the vector boson mass eigenvalues for each root in terms of B, up to an isoscalar factor that must be included for other than adjoint breaking.

In order to show that the boson mass eigenvalues are proportional to the weight-space scalar products of the roots and the symmetry breaking direction, we examine a simple example. The boson mass matrix in the tree approximation to the Higgs model has the form,

$$M^2_{\alpha\beta} = g^2 \sum_{\lambda,\lambda',\lambda''} \phi_V(\underline{r},\lambda')\langle \underline{r},\lambda' | X_\alpha | \underline{r},\lambda\rangle\langle \underline{r},\lambda | X_\beta | \underline{r},\lambda''\rangle \phi_V(\underline{r},\lambda''), \tag{16}$$

where the vacuum expectation value of the scalar fields $\phi_V(r,\lambda)$ has weight λ and belongs to representation $\underline{r}$, and $\langle \underline{r},\lambda | X_\alpha | \underline{r},\lambda'\rangle$ is a matrix element of the generator X_α. We use this notation in order to emphasize that $\phi_V(r,\lambda)$ is a tensor operator. Thus, we can use the commutation relations for tensor operators to rewrite (16) as

$$M^2_{\alpha\beta} = g^2 \sum_{\lambda} [X_\alpha , \phi_V(\underline{r},\lambda)][X_\beta , \phi_V(\underline{r},\lambda)] . \tag{17}$$

The advantage of writing the mass matrix in this basis-independent notation is that it is often quite simple to select a basis of the Lie algebra so that M^2 is diagonal.

Suppose that $\underline{r}$ is the adjoint representation of G and that there is sufficient gauge freedom to rotate $\phi_V(\underline{r},\lambda)$ into the Cartan subalgebra of G. This means that the vacuum expected value can be expanded as

$$\phi_V(\underline{r},\lambda) = \sum_{i=1}^{\ell} \tilde{c}_i H_i \tag{18}$$

The components $\tilde{c}'_i$ define the symmetry breaking direction in weight space.

The computation of the mass matrix requires a knowledge of the commutators, $[X_\alpha , H_i]$. This commutator is zero if X_α is in the Cartan subalgebra, so we can conclude immediately that ℓ vector bosons are massless. The mass matrix is diagonal if the remaining generators E_α are written in the Cartan-

Weyl basis, where the raising and lowering operators satisfy eigenvalue equations of the type,

$$[\, E_i \,,\; F_\alpha \,] = \alpha_i F_\alpha \quad , \tag{19}$$

where α_i is the i-th component of the root vector α. Upon substituting (19) into (17), we obtain

$$M^2_{\alpha\beta} = g^2 \mathrm{Tr}\left\{ (\alpha_i \tilde{c}_i) X_\alpha \, (\beta_j \tilde{c}_j) X_\beta \right\} = g^2 \delta_{\alpha\beta} \, (\alpha_i \tilde{c}_i)^2 , \tag{20}$$

where the last step implies a normalization convention. Thus, the mass eigenvalue of the vector boson associated with weight α is proportional to $(c, \alpha) = \tilde{c} \cdot \alpha$.

We return to the more general discussion. The next question is whether B is a direction defined by a vacuum expectation value with zero weight, or the direction is defined by nonzero weights. We show that B must have a contribution from zero weights. If B were due only to a nonzero weight, then the weight must be perpendicular to the electric charge axis, [illegible] not broken. The electric charge axis (5) is perpendicular to B [illegible], which, in turn, implies that the SU_5 lepto-diquarks receive [illegible] from B. The implication for the proton decay rate is obvious. For simplicity we can assume that the big breaking has zero weight. Then none of the bosons associated with the Cartan subalgebra get a superlarge mass, and we see from Table 1 that, at most, B can break U_6 to $SU_2 \times U_1 \times U_1 \times U_1 \times SU_3^c$.

The little breaking must have a component with $|\Delta I^W| = 1/2$. It can also have a component with $|\Delta I^W| = 0$, which has the same form as B in (15) if it has zero weight and is constrained by a = b + c if it has nonzero weight. For now we consider the $|\Delta I^W| = 1/2$ term only, which necessarily has nonzero weight, so each $|\Delta I^W| = 1/2$ weight must have the form,

$$\tilde{L} = [-d,\ d \pm 1,\ d + e,\ e,\ d \pm 1,\ 0] \quad . \tag{21}$$

Even without an explicit model of symmetry breaking, there are a few more comments that may prove interesting.

(1) Much of the discussion about B is not actually restricted to $\underline{78}$ breaking, except that the breaking representation must have triality zero. The only irreps of E_6 with zero weights are $\underline{78}$, $\underline{650}$, $\underline{2430}$, $\underline{2925}$, or larger.

(2) Suppose that B is due to an adjoint representation of Higgs scalars alone. The only independent Casimer invariants of E_6 are of order 2, 5, 6, 8, 9, and 12: thus the Higgs potential can depend on the length of the $\underline{78}$ only, and in the tree approximation there are no constraints on a, b, and c in (15). The one-loop corrections to the effective potential select a = 0 and b = -c, so an entire $SO_{10} \times U_1$ is left unbroken.[10] There is no reason to believe that, when bound states and other scalars are included, these radiative corrections would dominate the determination of B. In fact, it is conceivable that B is singled out by the weak breaking.

(3) If the adjoint breaking is along the only root with $|\Delta I^W| = 0$, which is (0-1 1 1-1-1), then an entire SU_6 is left unbroken.

(4) If the little breaking is in the $\underline{27}$, then there are three candidate $|\Delta I^W| = 1/2$ weights. Each of these breaks $SU_2 \times U_1 \times U_1 \times U_1$ to U_1^{em}.

(5) As in an SU_3 model with Higgs scalars transforming as $\underline{3} + \underline{\bar{3}} + \underline{8}$, it often happens that scalars in different irreps get vacuum values and, for a range of parameters, their directions are perpendicular in weight space. The weak breaking in the standard SU_5 model transforms as $\underline{5} + \underline{\bar{5}}$, or the $\underline{10}$ of the SO_{10} theory, which suggests that the weak breaking L has weights (0 1-1 0 1 0) and (-1 0 1-1 0 0) of the $\underline{27}$. This leads to a nice breaking pattern. L is perpendicular to B if a = 2c and b = 3c, so that B breaks E_6 to $SU_2 \times SU_2 \times U_1 \times U_1 \times SU_3^c$, and L breaks this on down to $U_1^{em} \times SU_3^c$. See Table 1.

3. AN E_6 MODEL WITH COMPOSITE MUON AND TAU FAMILIES

There is a widespread belief that the standard model of electromagnetic, weak, and strong interactions correctly describes low energy data. The phenomenological success of the $SU_2 \times U_1 \times SU_3{}^c$ model with left-handed doublet and right-handed singlet fermions is offset by the necessity of determining a large number of parameters and assigning a large number of elementary fermions from the analysis of a huge amount of experimental data. This difficulty is due partly to the semisimple group structure of the theory, but even more significantly for our considerations here, it is also due to the large number of known quarks and leptons.

The problems associated with the semisimple group structure may be overcome by embedding the standard model into a unifying group, which we denote by G'. The smallest candidate for G' is SU_5.[9] In this model the large number of known quarks and leptons are assigned to a highly reducible representation, and the fermions arranged in this way still appears disorderly. There have been several suggestions for tidying up the situation: (1) perhaps there is a local or other kind of family symmetry that is embedded together with SU_5 and possibly other factors into a yet larger unifying algebra; (2) perhaps, following the hints of supergravity, the fermions belong to a large irrep of a relatively small group; (3) perhaps none of the known quarks and leptons are elementary; or (4) perhaps some of the known fermions are elementary, (for example, one family), and the rest are composite.

We explore the last alternative here.[2] The proposal is not very radical and at first glance, it would not seem possible for that kind of model to give a satisfying account of the proliferation of elementary fermions. Nevertheless, there is an attractive example where the electron family (or muon family) is elementary and the muon (or electron family) and the tau family are composites. The binding force, color and flavor are unified into the exceptional group E_6[4] and the elementary fermions are assigned to the 27.

We now "derive" this E_6 model, because such a discussion shows that the model is quite unique. Suppose that G' is a simple Lie group that unifies color and flavor in the usual way,[5] and that there is a U_1 factor U_1^t, not in G', with a current that is coupled to a vector boson that provides the binding force.[11] We assume that $G' \times U_1^t$ is a maximal subgroup of a simple group G, so the elementary particle fields are assigned to irreps of G. The branching rules derived from the embedding,

$$G \supset G' \times U_1^t \quad , \tag{22}$$

provide (up to an overall scale) the binding charge eigenvalue of each irrep of G'.

The second assumption is the existence of a short-range attractive force between two particles with Q^t charges of opposite sign. The composite fermions are s-wave bound states of the elementary fermions and certain scalar particles in the theory. There are vector bosons with nonzero Q^t, but we assume that they get super heavy masses, and so are irrelevant for this discussion. An advantage of the fermion-scalar binding picture is the ease with which the fermion helicity structure is maintained. In addition to the composite fermions, there may also be composite scalars that are fermion-antifermion bound states.

The next assumptions are phenomenologically motivated. For parity to be conserved in the electromagnetic and strong interactions, the fermion assignments including the composite ones, must be vectorlike under color and electric charge. With regard to the weak interactions, we assume that parity is violated because the theory is flavor chiral, so the left-handed fermions are in a complex representation. In addition, we assume that there is one family of elementary fermions.

We select the G' that is as small as possible. G' must have complex irreps, so the smallest candidate for G' is SU_5, with the elementary fermions transforming as $\underline{\bar{5}} + \underline{10}$. Now, $SU_5 \times U_1^{\,t}$ is a maximal subgroup of SU_6, Sp_{10}, and SO_{10}. There are no irreps of dimension less than 5000 of SU_6 or Sp_{10} that contain a $\underline{\bar{5}} + \underline{10}$ of SU_5. SO_{10} is unacceptable because parity must be conserved in QED and QCD. The $\underline{\bar{5}}$ and $\underline{10}$ have different Q^t values, so the e^- and e^+ will bind to different scalars; similarly for d and $\bar{d}$.

We can avoid that difficulty by requiring that a left-handed fermion always be assigned to the same irrep of G' as its left-handed antiparticle image. This first happens for $SO_{10} \times U_1^{\,t}$, which is a maximal subgroup of SO_{12} and E_6. We discard SO_{12} because it gives a vectorlike theory for all interactions, even for the composite fermions. This leaves us with E_6 and the embedding,

$$E_6 \supset SO_{10} \times U_1^{\,t} \quad . \qquad (23)$$

The fundamental fields include the $\underline{78}$ vector bosons, the $\underline{27}$ of elementary fermions, and a $\underline{78}$ of scalars that can form bound states. The $\underline{78}$ scalars alone have no Yukawa couplings with the $\underline{27}$ of fermions, so there is at least an $E_6 \times E_6$ chiral invariance that may be broken dynamically by scalar bound states. This is a rather nice scenario, because, as we shall discuss, there is no gauge hierarchy problem.[12]

The eigenvalues of Q^t were computed in the last section. We list here the branching relations with the Q^t values in parenthesis:

$$\underline{27} = \underline{1}(4) + \underline{10}(-2) + \underline{16}(1) \qquad (24)$$

$$\underline{78} = \underline{1}(0) + \underline{45}(0) + \underline{16}(-3) + \overline{\underline{16}}(3) \qquad (25)$$

The normalization convention is set in Table 3.

We now use (24) and (25) to construct the bound state spectrum. We assume that neither the $\underline{27}$ nor $\underline{78}$ get superlarge masses, but are heavy enough and the binding strong enough that the composites appear pointlike to all probes made so far, such as in lepton pair production or g-2 experiments. The composite fermions are due to the binding of the $\underline{16}$ and $\overline{\underline{16}}$ in the $\underline{78}$ to the elementary fermions with Q^t of opposite sign. The spectrum of left-handed fermions, classified by SO_{10} irreps is

$$f_L = (\underline{1} + \underline{10} + \underline{16}) + (\underline{16}) + (\underline{16} + \overline{\underline{144}}) \quad . \qquad (26)$$

The first set in (26) are the elementary fermions in the $\underline{27}$; the second set arises from the binding of the SO_{10} singlet fermion to the $\underline{16}$ of scalars, and is the most tightly bound; the third set arises from the binding of the $\underline{10}$ of fermions to the $\overline{\underline{16}}$ of scalars; and a fourth set that is omitted from (26) would be due to the binding of $\underline{16}$ and $\underline{16}$, which would give a set with $\underline{10} + \underline{120} + \underline{126}$. This last set is least strongly bound, if bound at all, and we shall neglect it for the remainder of the talk. A similar discussion of the scalar bound states in $\underline{27} \times \underline{27}$ is also possible, and these might be used to break the weak interaction group dynamically. The composite weights in the $\overline{\underline{27}}$ coincide with the example at the end of Sec. 2.

Perhaps the most amusing feature of (26) is the occurance of three families of $\underline{16}$'s The electron and muon families can be assigned to the first two $\underline{16}$'s; without further analysis it is not possible to decide which is elementary. The τ family should be either in the third $\underline{16}$ or in the $\overline{\underline{144}}$, which also contains a $\overline{\underline{5}} + \underline{10}$ of SU_5. Since the $\underline{16}$ and $\overline{\underline{144}}$ are bound with the same overall binding strength, a calculation is needed to decide which has the lowest mass states. What is more significant is that this model predicts a great proliferation of quarks, leptons, and other fermions not too far above the τ and b masses. Although this is not a unique prediction of this bound state model, it will be interesting to see what will be discovered above present PETRA energies.

For the composite fermion model to imitate physical reality, the binding force must be very strong and very short ranged. Thus the vector boson mediating the force must be very heavy (perhaps around 10 TeV) and the coupling $\alpha_t(Q^2)$ must be very large at small Q^2. Without adequate field theory technology to compute large running couplings, it is not possible to give quantitative results about the masses and the behavior of the binding force. However, the behavior of small couplings is well understood from the perturbation theoretic treatment of the renormalization group equations; since we know that the values of the standard model gauge couplings are small at Q^2 around 10 GeV^2, it is necessary to check whether it is indeed possible to compute $\alpha_t(Q^2)$ for all Q^2 .

The argument that the gauge couplings do become large is one of self consistency. If the $\underline{78}$'s of vector bosons and of scalars and the $\underline{27}$ of fermions are the only contributions to the one loop apprximation to the running coupling constant equations, then the theory is asymptotically free. (We ignore the scalar self couplings in this consideration.) Thus, if the strong coupling is small at 10 GeV^2, then it is even smaller at larger Q^2, and there is no reason to believe that any of the E_6 couplings get large. However, if the composite states are tightly enough bound that they also contribute to the one loop approximation over, say, the range of 100 GeV to 10 TeV, then they can destroy the asymptotic freedom of QCD and can push the coupling up into a region where we cannot, at present, compute it.

There are 41 effective flavors of quarks in (26), and in the one-loop approximation, we can estimate the value of $\bar{Q}^2$, where the QCD coupling becomes large. It is

$$\sqrt{\bar{Q}^2} = \sqrt{Q^2}\exp\left[-\left(\frac{3\pi}{\alpha_t(Q^2)}\right)\left(\frac{1}{33-2n_3-10n_6-12n_8}\right)\right] = \sqrt{Q^2}\exp\left[0.19/\alpha_t(Q^2)\right], \tag{27}$$

where "33" is the gluon contribution, n_3 is the number of Dirac quarks, n_6 is the number of Dirac $\underline{6}$'s, and n_8 is the number of Dirac octets. This rather rapid growth makes it possible to speculate that in a region between the weak boson mass and the unification mass, the gauge couplings may be large and $\alpha_t(Q^2)$ "freezes out" with a large value. We could also speculate that the couplings do eventually become small and asymptotically free before unification, (at large Q^2 the bound states no longer contribute) so that the proton decay rate is not too fast and the unification mass is not too close to the Planck mass. The dynamical symmetry breaking may cause the desert to bloom.

If this scenario is correct, then we must change attitudes toward some computations in unified models. Problems and advantages seem to be reversed over the situation with the usual SU_5 model. The unification mass, weak mixing angle, and quark masses are not easily computed, because the perturbation theory formulas do not hold over the whole extrapolation range. Thus there is no calculation of the proton lifetime, although this is not a difficulty of principle. The gauge hierarchy problem takes on a new character in the composite fermion model. Suppose the superstrong breaking is due to explicit Higgsism. Then at the unification scale, the composite scalars that do the weak breaking do not exist. The composite scalars appear elementary only on the scale of 100 GeV, and then may be available to do the weak breaking. Thus, the hierarchy problem would be resolved if the weak breaking were due to this dynamical mechanism. Further details of this model can be found in Ref. 2.

ACKNOWLEDGMENTS

I wish to thank Gordon Shaw for a very pleasant collaboration and Pierre Ramond for many helpful conversations.

References.

1. E. B. Dynkin, Amer. Math. Soc. Trans. Ser. 2, 6, 111 and 245 (1957).
2. G. L. Shaw and R. Slansky, Los Alamos Preprint (1980).
3. B. G. Wybourne, "Classical Groups for Physicists", Wiley-Interscience, New York (1974).
4. F. Gürsey, P. Ramond, and P. Sikivie, Phys. Lett. B60, 177 (1975).
5. M. Gell-Mann, P. Ramond, and R. Slansky, Rev. Mod. Phys. 50, 721 (1978).
6. W. McKay, J. Patera, and D. Sankoff, "Computers in Non Associative Rings and Algebras", ed. R. Beck and B. Kolman, Academic Press, New York (1977), p. 235.
7. W. McKay and J. Patera, "Tables of Dimensions, Second and Fourth Indices and Branching Rules of Simple Lie Algebras", M. Dekker Inc., New York (1980); J. Patera and D. Sankoff, "Tables of Branching Rules for Representations of Simple Lie Algebras", L'Université de Montréal, Montréal (1973).
8. R. Slansky, in preparation; P. Ramond, Caltech preprint CALT-68-945 (1980).
9. H. Georgi and S. L. Glashow, Phys. Rev. Lett. 32, 438 (1974), and H. Georgi, H. Quinn, and S. Weinberg, Phys. Rev. Lett. 33, 451 (1974).
10. J. Harvey, Caltech preprint CALT-68-737 (1979).
11. G. L. Shaw, Phys. Lett. 81B, 343 (1979).
12. E. Gildener, Phys. Rev. D14, 1667 (1976), and S. Weinberg, Phys. Lett. 82B, 387 (1979).

Table 1. Nonzero E_6 Roots.

root	level	color	Q^{em}	I_3^w	Q^t	$SU_5(SO_{10})$	$\tilde{B}\cdot\alpha$	$\tilde{L}\cdot\alpha$
Color SU_3 roots								
(0 0 0 0 0 1)	0	(1 1)	0	0	0	24(45)	0	0
(0 1 0 0-1 0)	4	(2-1)	0	0	0	24(45)	0	0
(0-1 0 0 1 1)	7	(-1 2)	0	0	0	24(45)	0	0
Left-handed SU_3 roots								
(1 0 0 0 1-1)	6	(0 0)	1	1	0	24(45)	0	±1
(-1 1 0 0 1-1)	7	(0 0)	0	1/2	-3	$\bar{5}$(16)	3c	3d ± 2
(-2 1 0 0 0 0)	12	(0 0)	-1	-1/2	-3	$\bar{5}$(16)	3c	3d ± 1
Right-handed SU_3 roots								
(0-1 1 1-1-1)	9	(0 0)	0	0	3	1($\overline{16}$)	a+b-2c	-d+2e ∓ 2
(0 0-1 2-1 0)	10	(0 0)	-1	0	3	$\overline{10}$($\overline{16}$)	-a+2b-c	-2d+e ∓ 1
(0-1 2-1 0-1)	10	(0 0)	1	0	0	10(45)	2a-b-c	d+e ∓ 1
SU_5 and lepto-diquarks								
(1-1 1-1 1 0)	4	(0 1)	4/3	1/2	0	24(45)	a-b-c	0
(1 0 1-1 0-1)	8	(1-1)	4/3	1/2	0	24(45)	a-b-c	0
(1-1 1-1 1-1)	15	(-1 0)	4/3	1/2	0	24(45)	a-b-c	0
(0-1 1-1 0 1)	9	(0 1)	1/3	-1/2	0	24(45)	a-b-c	∓ 1
(0 0 1-1-1 0)	13	(1-1)	1/3	-1/2	0	24(45)	a-b-c	∓ 1
(0-1 1-1 0 0)	20	(-1 0)	1/3	-1/2	0	24(45)	a-b-c	∓ 1
SO_{10}/SU_5 leptoquarks								
(0 0 1 0 0-1)	1	(1 0)	2/3	1/2	0	10(45)	a	d+e
(0-1 1 0 1-1)	8	(-1 1)	2/3	1/2	0	10(45)	a	d+e
(0 0 1 0 0-2)	12	(0-1)	2/3	1/2	0	10(45)	a	d+e
(-1 0 1 0-1 0)	6	(1 0)	-1/3	-1/2	0	10(45)	a	d+e ∓ 1
(-1-1 1 0 0 0)	13	(-1 1)	-1/3	-1/2	0	10(45)	a	d+e ∓ 1
(-1 0 1 0-1-1)	17	(0-1)	-1/3	-1/2	0	10(45)	a	d+e ∓ 1
(-1 0 0 1 0 0)	4	(0 1)	-2/3	0	0	10(45)	b+c	d+e
(-1 1 0 1-1-1)	8	(1-1)	-2/3	0	0	10(45)	b+c	d+e
(-1 0 0 1 0-1)	15	(-1 0)	-2/3	0	0	10(45)	b+c	d+e
E_6/SO_{10} leptoquarks								
(0 1 0-1 1 0)	3	(1 0)	2/3	1/2	-3	10(16)	-b+2c	2d-e ± 2
(0 0 0-1 2 0)	10	(-1 1)	2/3	1/2	-3	10(16)	-b+2c	2d-e ± 2
(0 1 0-1 1-1)	14	(0-1)	2/3	1/2	-3	10(16)	-b+2c	2d-e ± 2

(continued next page)

Table 1. (continued)

(-1 1 0-1 0 1)	8	(1 0)	-1/3	-1/2	-3	10(16)	-b+2c	2d-e ± 1
(-1 0 0-1 1 1)	15	(-1 1)	-1/3	-1/2	-3	10(16)	-b+2c	2d-e ± 1
(-1 1 0-1 0 0)	19	(0-1)	-1/3	-1/2	-3	10(16)	-b+2c	2d-[illegible] ± 1
(-1 0 1-1 1 0)	5	(0 1)	1/3	0	-3	$\bar{5}$(16)	a-b+2c	3d ± 1
(-1 1 1-1 0-1)	9	(1-1)	1/3	0	-3	$\bar{5}$(16)	a-b+2c	3d ± 1
(-1 0 1-1 1-1)	16	(-1 0)	1/3	0	-3	$\bar{5}$(16)	a-b+2c	3d ± 1
(-1 1-1 0 1 1)	6	(0 1)	-2/3	0	-3	10(16)	-a+3c	2d-e ± 2
(-1 2-1 0 0 0)	10	(1-1)	-2/3	0	-3	10(16)	-a+3c	2d-e ± 2
(-1 1-1 0 1 0)	17	(-1 0)	-2/3	0	-3	10(16)	-a+3c	2d-e ± 2

Table 2. Weights and Content of the <u>27</u> of E_6 .

weight	level	color	Q^{em}	I_3^w	Q^t	$SU_5(SO_{10})$	SO_{10} weight
(0 0 0 1 0-1)	4	(0 0)	0	1/2	1	$\bar{5}$(16)	(1-1 0 1 0)
(-1 0 0 1-1 0)	9	(0 0)	-1	-1/2	1	$\bar{5}$(16)	(1 0 0 0-1)
(1-1 1-1 0 0)	9	(0 0)	1	0	1	10(16)	(-1 0 1-1 0)
(1 0-1 0 0 1)	10	(0 0)	0	0	1	1(16)	(-1 1-1 0 1)
(0 0 1-1 1-1)	5	(0 0)	1	1/2	-2	5(10)	(0-1 1 0 0)
(-1 0 1-1 0 0)	10	(0 0)	0	-1/2	-2	5(10)	(0 0 1-1-1)
(0 1-1 0 1 0)	6	(0 0)	0	1/2	-2	$\bar{5}$(10)	(0 0-1 1 1)
(-1 1-1 0 0 1)	11	(0 0)	-1	-1/2	-2	$\bar{5}$(10)	(0 1-1 0 0)
(1-1 0 1-1 0)	8	(0 0)	0	0	4	1(1)	(0 0 0 0 0)
(1 0 0 0 0 0)	0	(1 0)	2/3	1/2	1	10(16)	(0 0 0 0 1)
(1-1 0 0 1 0)	7	(-1 1)	2/3	1/2	1	10(16)	(-1 0 0 1 0)
(1 0 0 0 0-1)	11	(0-1)	2/3	1/2	1	10(16)	(0-1 0 0 1)
(0 0 0 0-1 1)	5	(1 0)	-1/3	-1/2	1	10(16)	(0 1 0-1 0)
(0-1 0 0 0 1)	12	(-1 1)	-1/3	-1/2	1	10(16)	(-1 1 0 0-1)
(0 0 0 0-1 0)	16	(0-1)	-1/3	-1/2	1	10(16)	(0 0 0-1 0)
(-1 1 0 0 0 0)	1	(1 0)	-1/3	0	-2	5(10)	(1 0 0 0 0)
(-1 0 0 0 1 0)	8	(-1 1)	-1/3	0	-2	5(10)	(0 0 0 1-1)
(-1 1 0 0 0-1)	12	(0-1)	-1/3	0	-2	5(10)	(1-1 0 0 0)
(0 0 0-1 1 1)	4	(0 1)	1/3	0	-2	$\bar{5}$(10)	(-1 1 0 0 0)
(0 1 0-1 0 0)	8	(1-1)	1/3	0	-2	$\bar{5}$(10)	(0 0 0-1 1)
(0 0 0-1 1 0)	15	(-1 0)	1/3	0	-2	$\bar{5}$(10)	(-1 0 0 0 0)
(0-1 1 0 0 0)	2	(0 1)	1/3	0	1	$\bar{5}$(16)	(0 0 1 0-1)
(0 0 1 0-1-1)	6	(1-1)	1/3	0	1	$\bar{5}$(16)	(1-1 1-1 0)
(0-1 1 0 0-1)	13	(-1 0)	1/3	0	1	$\bar{5}$(16)	(0-1 1 0-1)
(0 0-1 1 0 1)	3	(0 1)	-2/3	0	1	10(16)	(0 1-1 1 0)
(0 1-1 1-1 0)	7	(1-1)	-2/3	0	1	10(16)	(1 0-1 0 1)
(0 0-1 1 0 0)	14	(-1 0)	-2/3	0	1	10(16)	(0 0-1 1 0)

Table 3. Physical Roots and Axes in E_6 Weight Space.

	Dynkin Basis	Dual Basis
Color Roots	(0 0 0 0 0 1)	[1 2 3 2 1 2]
	(0 1 0 0-1 0)	[1 2 2 1 0 1]
	(0-1 0 0 1 1)	[0 0 1 1 1 1]
Weak Isospin Root	(1 0 0 0 1-1)	[1 1 1 1 1 0]
Q^{em} axis	$\frac{1}{3}$(3-2 3-3 2-2)	$\frac{1}{3}$[2 1 2 0 1 0]
$I_3{}^w$ axis	$\frac{1}{2}$(1 0 0 0 1-1)	$\frac{1}{2}$[1 1 1 1 1 0]
Y^w axis	$\frac{1}{3}$(3-4 6-6 1-1)	$\frac{1}{3}$[1-1 1-3-1 0]
Q^t axis	3(1-1 0 1-1 0)	[1-1 0 1-1 0]
Q^r axis	(-3-1 4 1-1-4)	[-1 1 4 3 1 0]
B axis	(-3c, 3c-a, 2a-b-c, 2b-a-c, 2c-b, -a)	[-c c a b c 0]
L axis	$(-3d \mp 1, 2d-e \pm 2, d+e \mp 1, e-2d \mp 1, 2d-e \pm 2, d+e)$	$[-d, d \pm 1, d+e, e, d \pm 1, 0]$